Turbo: Design of a Constraint Programming Solver on GPU

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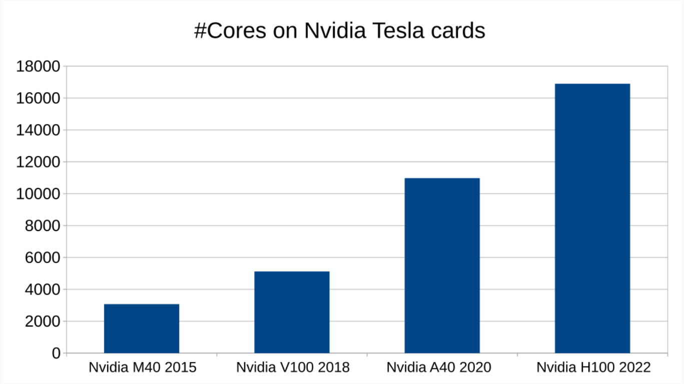
Why CP on GPU?

CPU clock speed is stagnating, GPU #cores is increasing quickly each year.



Easy speed-up: same code but faster.

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"The biggest lesson that can be read from 70 years of AI research is that general methods that leverage computation are ultimately the most effective, and by a large margin." [a]

[a] The Bitter Lesson, Rich Sutton, 2019, `http://www.incompleteideas.net/IncIdeas/BitterLesson.html`

Easy speed up: same code but faster

Why CP on GPU?

"Leading LLM researchers are already espousing Rich Sutton's bitter lesson; the fact that almost no innovation is required beyond scale."[a]

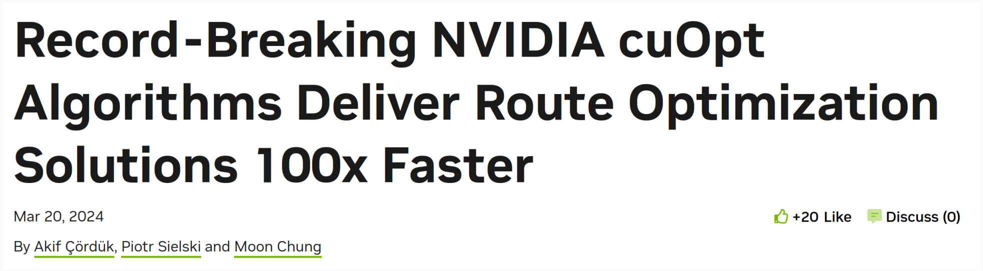
[a] https://medium.com/@felixhill/200bn-weights-of-responsibility-da85a44a2c5e

Why CP on GPU?

- Machine learning (deep learning, reinforcement learning, . . .) has seen tremendous speed-ups (e.g. 100x, 1000x) by using GPU.
- Some (sequential) optimizations on CPU are made irrelevant if we can explore huge state space faster.

Can we replicate the success of machine learning applications to combinatorial optimization?

cuOpt: Nvidia Routing Optimization



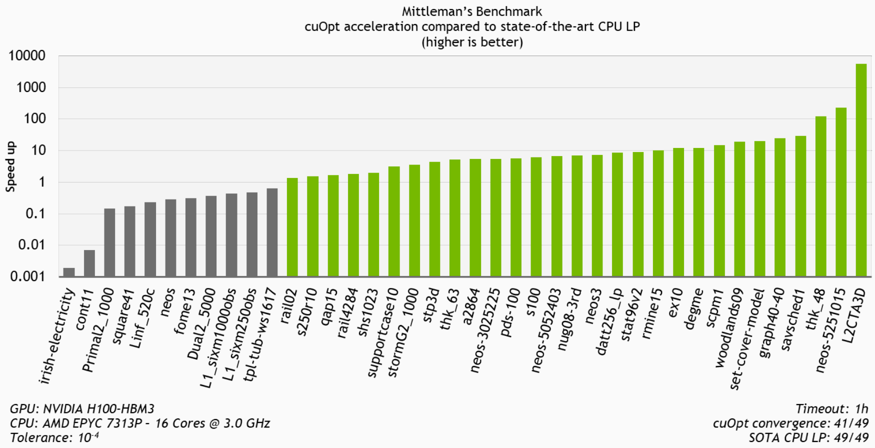
Record-Breaking NVIDIA cuOpt Algorithms Deliver Route Optimization Solutions 100x Faster

Mar 20, 2024

By Akif Çördük, Piotr Sielski and Moon Chung

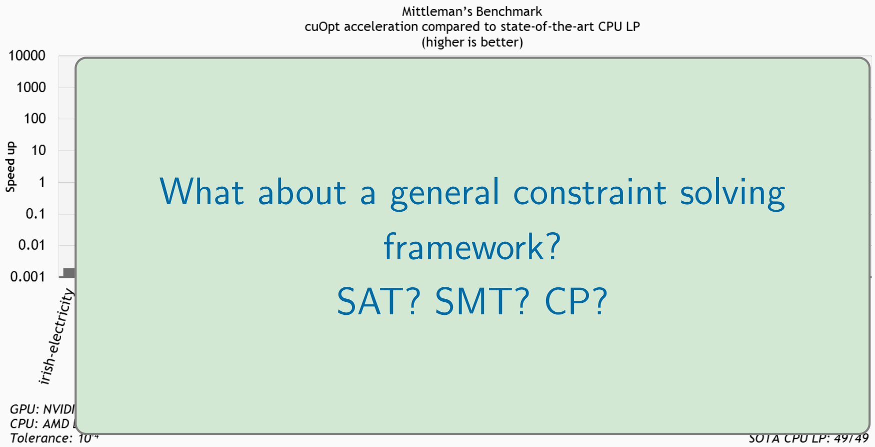
Cool, but specialized combinatorial algorithm for routing.

cuOpt: Nvidia Linear Programming Solver



Better! But only for linear constraints over continuous domain.

cuOpt: Nvidia Linear Programming Solver



What about a general constraint solving framework?
SAT? SMT? CP?

Better! But only for linear constraints over continuous domain.

Combinatorial Optimization on GPU

Very scarce literature, usually:

- **Heuristics**: often population-based algorithms[1].
- **Limited set of problems**[2]
- **Limited GPU parallelization**: offloading to GPU specialized filtering procedures[3,4].
- **Limited expressivity**: solver with max 256 set variables[5].

For CP-based approach: no code and no proof of correctness.

[1] A. Arbelaez and P. Codognet, *A GPU Implementation of Parallel Constraint-Based Local Search*, PDP, 2014.
[2] Jan Gmys. Exactly Solving Hard Permutation Flowshop Scheduling Problems on Peta-Scale GPU-Accelerated Supercomputers. INFORMS Journal on Computing, 2022.
[3] F. Campeotto et al., *Exploring the use of GPUs in constraint solving*, PADL, 2014
[4] F. Tardivo et al., *Constraint propagation on GPU: A case study for the AllDifferent constraint*, Journal of Logic and Computation, 2023.
[5] A. Dovier et al., *CUDA: Set Constraints on GPUs*, 2022.

Our Contributions

Theoretical Foundation

We propose a new parallel model of computation:

- **Correct**: formal proofs of correctness w.r.t. the parallel load/store operations on memory[6].
- **Simple and lock-free**: no mutex, no complicated atomic primitives, just barriers.
- **Expressive**: a process algebra based on *concurrent constraint programming* which supports $\mathbb{Z}, \mathbb{R}, \mathcal{P}(\mathbb{Z}), \ldots$ domains.

[6] P. Talbot et al., *A Variant of Concurrent Constraint Programming on GPU*, AAAI, 2022.

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Implementation

TURBO: a constraint solver fully executing on GPU.

- **General**: Support MiniZinc and XCSP3 constraint models (currently only $\mathbb{Z}$ variables).
- **Open-source**: Publicly available on `https://github.com/ptal/turbo`.
- **Efficient?**: In progress! $\times 10$ in 1 year (`https://lattice-land.github.io/`).

[6] P. Talbot et al., *A Variant of Concurrent Constraint Programming on GPU*, AAAI, 2022.

What is in this presentation?

On the menu:

1. Constraint Programming (CP)
2. Graphical Processing Units (GPU)
3. Towards a **correct** GPU constraint solver.
4. Towards an **efficient** GPU constraint solver.

Constraint Programming

Constraint Programming

Constraint satisfaction problem (CSP)

A CSP is a pair $\langle d, C\rangle$, example:

$$\langle\{x \mapsto [1,10], y \mapsto [1,10], z \mapsto [1,10]\}, \{x+3 \leq y, x+6 \leq z\}\rangle$$

A solution is $\{x \mapsto 1, y \mapsto 4, z \mapsto 9\}$.

Notation: given $d(x) = [a, b]$, we write $d(x)_\ell = a$ and $d(x)_u = b$.

Propagators

Propagator for the constraint $x + k \leq y$ with x, y variables and $k \in \mathbb{Z}$ a constant:

$$\begin{aligned}
&\mathit{propagate}(d, x + k \leq y) \triangleq \\
&\qquad \text{update } d(x)_u \text{ to } d(y)_u - k \\
&\qquad \text{update } d(y)_\ell \text{ to } d(x)_\ell + k
\end{aligned}$$

Definition

A propagator is a *reductive* ($f(x) \leq x$) and *monotone* ($x \leq y \Rightarrow f(x) \leq f(y)$) function over d.

Example

Memory:

$x = [1, 10]$
$y = [1, 10]$
$z = [1, 10]$

Propagators:

$propagate(d, x + 3 \leq y) =$
update $d(x)_u$ to $d(y)_u - 3$
update $d(y)_\ell$ to $d(x)_\ell + 3$

$propagate(d, x + 6 \leq z) =$
update $d(x)_u$ to $d(z)_u - 6$
update $d(z)_\ell$ to $d(x)_\ell + 6$

Example

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$propagate(d, x + 6 \leq z) =$

update $d(x)_u$ to $d(z)_u - 6$

update $d(z)_\ell$ to $d(x)_\ell + 6$

Example

Memory:

$x = [1, 7]$
$y = [1, 10]$
$z = [1, 10]$

Propagators:

$propagate(d, x + 3 \leq y) =$
    update $d(x)_u$ to $d(y)_u - 3$
    update $d(y)_\ell$ to $d(x)_\ell + 3$

$propagate(d, x + 6 \leq z) =$
    update $d(x)_u$ to $d(z)_u - 6$
    update $d(z)_\ell$ to $d(x)_\ell + 6$

Example

Propagators:

Memory:

$x = [1, 7]$
$y = [4, 10]$
$z = [1, 10]$

$propagate(d, x + 3 \leq y) =$

update $d(x)_u$ to $d(y)_u - 3$

update $d(y)_\ell$ to $d(x)_\ell + 3$

$propagate(d, x + 6 \leq z) =$

update $d(x)_u$ to $d(z)_u - 6$

update $d(z)_\ell$ to $d(x)_\ell + 6$

Example

Memory:

$x = [1, 4]$
$y = [4, 10]$
$z = [1, 10]$

Propagators:

$propagate(d, x + 3 \leq y) =$
  update $d(x)_u$ to $d(y)_u - 3$
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Fixpoint

Memory:

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update $d(z)_\ell$ to $d(x)_\ell + 6$

Fixpoint

- There is no propagator that can update the domains!
- We have reached a **fixpoint**: $f(x) = x$.

But... the fixpoint of propagation is not necessarily a solution

Memory:

$x = [1, 4]$
$y = [4, 10]$
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Propagators:

$propagate(d, x + 3 \leq y) =$
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But... the fixpoint of propagation is not necessarily a solution

Memory:

$x = [1, 4]$
$y = [4, 10]$
$z = [7, 10]$

The result is not a solution yet!

$$\{x \mapsto 4, y \mapsto 4, z \mapsto 7\}$$

$$x + 3 \leq y$$

Initial problem

$x = [1, 4]$
$y = [4, 10]$
$z = [7, 10]$

Subproblem 1 $(x = 1)$

$x = [1, 1]$
$y = [4, 10]$
$z = [7, 10]$

Solution reached!

$\{x \mapsto [1, 1], y \mapsto [4, 10], z \mapsto [7, 10]\}$

Subproblem 2 $(x > 1)$

$x = [2, 4]$
$y = [4, 10]$
$z = [7, 10]$

Not a solution, need to be further split into more subproblems.

A constraint solving algorithm: propagate and search

Let $\langle d, \{c_1, \ldots, c_n\}\rangle$ be a CSP.

- **Propagate**: Remove inconsistent values from the variables' domain: $\mathit{propagate}(d, \{c_1, \ldots, c_n\}) \triangleq \mathbf{gfp}\ ((\lambda x.x \sqcap d) \circ p_1 \circ \ldots \circ p_n)$ with $p_i \triangleq \mathit{propagate}(d, c_i)$.
- **Search**: Divide the problem into (complementary) subproblems explored using *backtracking*.

```
solve($\langle d, C\rangle$) =
    $d' \leftarrow$ propagate($\langle d, C\rangle$)
    if $d'$ is a solution then
        return $\{d'\}$
    else if $d'$ has an empty domain then
        return $\{\}$
    else
        $\langle d_1, \ldots, d_n\rangle \leftarrow$ branch($d'$)
        return $\bigcup_{i=0}^{n}$ solve($\langle d_i, C\rangle$)
    end if
```

A constraint solving algorithm: propagate and search

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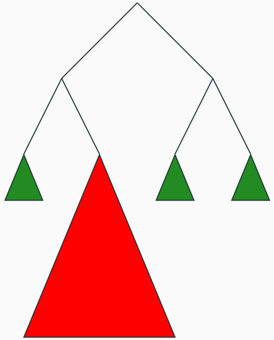
```
$\texttt{solve}(\langle d, C\rangle) =$
  $d' \leftarrow \texttt{propagate}(\langle d, C\rangle)$
  **if** $d'$ is a solution **then**
    **return** $\{d'\}$
  **else if** $d'$ has an empty domain **then**
    **return** $\{\}$
  **else**
    $\langle d_1, \ldots, d_n\rangle \leftarrow \texttt{branch}(d')$
    **return** $\bigcup_{i=0}^{n} \texttt{solve}(\langle d_i, C\rangle)$
  **end if**
```

How to parallelize propagate and search?

Parallel CP (on CPU)

Parallel Search Made Simple[7]

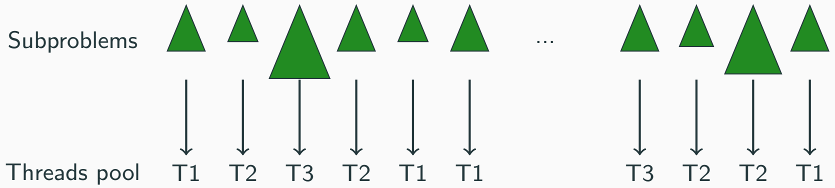


Unbalanced tree: use work-stealing but then threads need to communicate (complicated data structure, efficiency loss, ...).

[7] C. Schulte, ‘Parallel search made simple’, in Proceedings of TRICS, 2000

Embarrasingly Parallel Search (EPS)[8]

- **Idea**: Divide the problem into many subproblems beforehand (e.g. $N \times 30$ with N the number of threads).
- **Intuition**: Statistically, there is little chance a subproblem takes longer than the sum of the other subproblems.



[8]A. Malapert, J.-C. Régin, and M. Rezgui, 'Embarrassingly Parallel Search in Constraint Programming', JAIR, 2016

What About Parallel Propagation?

State of the art

- Parallelization of propagation has never worked well on CPU[9].
- It was not (and still not) very successful in Prolog (AND-Parallelism)[10].

$\Rightarrow$ There is no solver using parallel propagation on CPU!

WHY ?

[9] I. P. Gent et al., *A review of literature on parallel constraint solving*, Theory and Practice of Logic Programming, 2018.
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WHY ?

- OR-parallelism is easy: independent subproblems per core (**almost linear scaling**).
- AND-parallelism is hard: propagators works on the same data.

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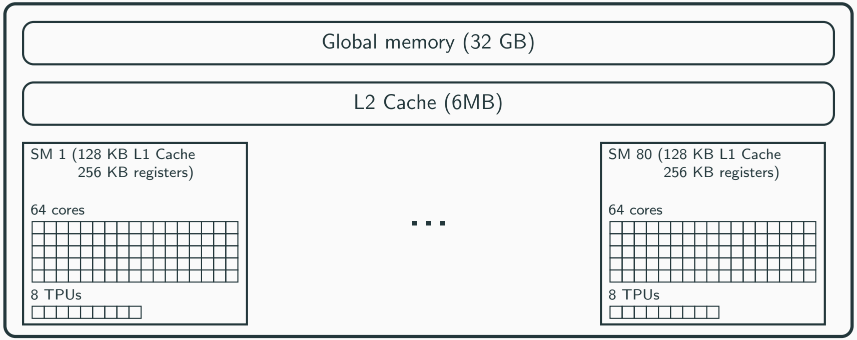
GPU Architecture

(Simplified) Architecture of the GPU Nvidia V100


Global memory (32 GB)
L2 Cache (6MB)
SM 1 (128 KB L1 Cache 256 KB registers)
64 cores
8 TPUs
...
SM 80 (128 KB L1 Cache 256 KB registers)
64 cores
8 TPUs

5120 cores on a single V100 GPU @ 1290MHz

640 Tensor Processing Units (TPUs)

Whitepaper: `https://images.nvidia.com/content/volta-architecture/pdf/volta-architecture-whitepaper.pdf`

Programming Challenges

- **Memory coalescence**: the way to access the data is important (factor 10).
- **Thread divergence**: each thread within a warp (group of 32 threads) should execute the same instructions.
- **Memory allocation** (dynamic data structures): costly on GPU, everything is generally pre-allocated.
- **Other limitations**: small cache, limited number of lines of code, ...

Example: Find the Minimum in an Array

Each thread computes its local min (*map*), then we compute the min of all local min (*reduce*).

- Map:

$$\underbrace{\boxed{3}\boxed{22}\boxed{10}\boxed{23}}_{\text{Thread 0, } m_0 = 3}\underbrace{\boxed{21}\boxed{7}\boxed{91}\boxed{1}}_{\text{Thread 1, } m_1 = 1}\underbrace{\boxed{3}\boxed{10}\boxed{42}\boxed{11}}_{\text{Thread 2, } m_2 = 3}\underbrace{\boxed{8}\boxed{7}\boxed{32}}_{\text{Thread 3, } m_3 = 7}$$

- Reduce: $min([3, 1, 3, 7]) = 1$.

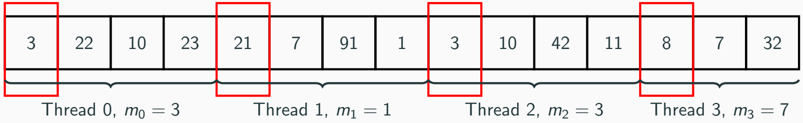
Example: Find the Minimum in an Array

Intuitive implementation

```
__global__ void parallel_min(int* v, size_t n, int* local_min) {
  local_min[threadIdx.x] = INT_MAX;
  size_t m = n / blockDim.x + (n % blockDim.x != 0);
  size_t from = threadIdx.x * m;
  size_t to = min(n, from + m);
  for(size_t i = from; i < to; ++i) {
    local_min[threadIdx.x] = min(local_min[threadIdx.x], v[i]);
  }
}
```

Iteration 1:

- Map:



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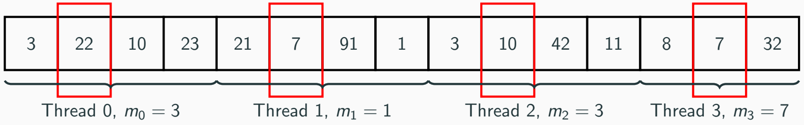
Iteration 2:

- Map:



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  for(size_t i = from; i < to; ++i) {
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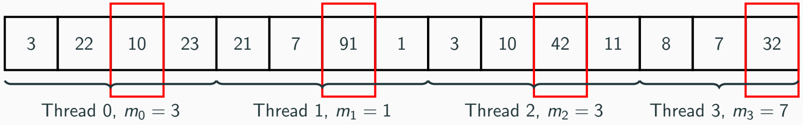
}
```

Iteration 3:

- Map:



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  }
}
```
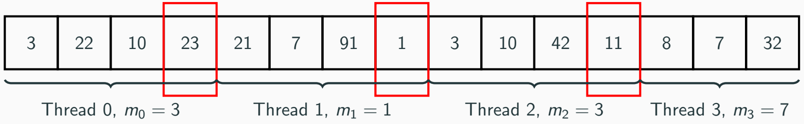

Iteration 4:

- Map:



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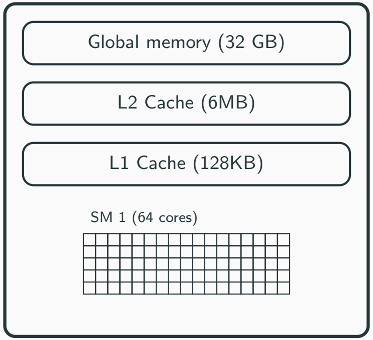
Optimization: Coalesced Memory Accesses

Knowing about the hardware is crucial for efficiency.

- Previous implementation can work well on CPU since each core has its own cache.
- On GPU, it is better to access the memory contiguously—it allows the GPU to move data from global memory to cache faster.



Optimization: Coalesced Memory Accesses

Strided implementation

```
__global__ void parallel_min_stride(int* v, size_t n, int* local_min) {
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  for(size_t i = threadIdx.x; i < n; i += blockDim.x) {
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  }
}
```

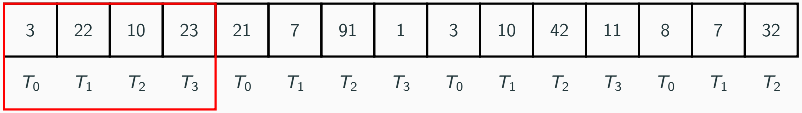

Very important: up to an order of magnitude faster (10x).

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```

3	22	10	23	21	7	91	1	3	10	42	11	8	7	32
T_0	T_1	T_2	T_3	T_0	T_1	T_2	T_3	T_0	T_1	T_2	T_3	T_0	T_1	T_2

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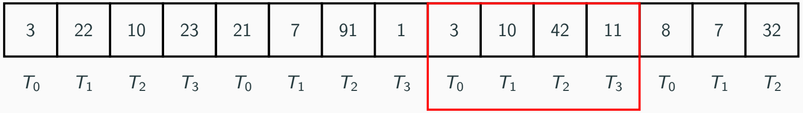
3 22 10 23 21 7 91 1 3 10 42 11 8 7 32
T_0 T_1 T_2 T_3 T_0 T_1 T_2 T_3 T_0 T_1 T_2 T_3 T_0 T_1 T_2

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  local_min[threadIdx.x] = INT_MAX;
  for(size_t i = threadIdx.x; i < n; i += blockDim.x) {
    local_min[threadIdx.x] = min(local_min[threadIdx.x], v[i]);
  }
}
```

3	22	10	23	21	7	91	1	3	10	42	11	8	7	32
T_0	T_1	T_2	T_3	T_0	T_1	T_2	T_3	T_0	T_1	T_2	T_3	T_0	T_1	T_2

Very important: up to an order of magnitude faster (10x).

Towards a Correct GPU Constraint Solver

Architecture Overview

- **Each GPU block solves an independent subproblem**: similar to embarassingly parallel search on CPU.
- Why not 1 subproblem per GPU thread? $\Rightarrow$ not enough cache per thread.

How to parallelize the propagation inside a GPU block?

Parallel Concurrent Constraint Programming (PCCP)

History

Concurrent constraint programming (CCP) is a *process calculus* introduced in the eighties[11].
Two main operations: ask and tell (example on whiteboard).

But CCP lacked a proper connection to parallel architecture, and we worked on that by simplifying the language (no recursion)[12].

[11] V. A. Saraswat and M. Rinard, *Concurrent constraint programming* (POPL-89)
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Syntax of PCCP

Let $x, y, y_1, \dots, y_n \in Vars$ be variables, L a lattice, f a monotone function, and b a Boolean variable of type $\langle\{true, false\}, \Leftarrow\rangle$:

$$\begin{array}{lll} \langle P, Q\rangle ::= & \texttt{if}\ \ b\ \ \texttt{then}\ \ P & \textit{ask statement} \\ \mid & x \leftarrow f(y_1, \dots, y_n) & \textit{tell statement} \\ \mid & \exists x{:}\mathrm{L},\ P & \textit{local statement} \\ \mid & P \parallel Q & \textit{parallel composition} \end{array}$$

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Denotational Semantics

- A PCCP process is a reductive and monotone function over a Cartesian product $\mathit{Store} = L_1 \times \ldots \times L_n$ storing the values of all local variables.
- Since we do not have recursion, we know at compile-time the number of variables.
- Let *Proc* be the set of all PCCP processes.

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Denotational Semantics

We define a function $\mathcal{D} : \mathit{Proc} \to (\mathit{Store} \to \mathit{Store})$:

$$\begin{aligned}
&\mathcal{D}(x \leftarrow f(y_1, .., y_n)) \triangleq \lambda s.s[x \mapsto s(x) \sqcap f(s(y_1), .., s(y_n))] \\
&\mathcal{D}(\texttt{if } b \texttt{ then } P) \triangleq \lambda s.(\!|\ s(b) \,?\, \mathcal{D}(P)(s) \,\text{:}\, s\ |\!) \\
&\mathcal{D}(P \parallel Q) \triangleq \mathcal{D}(P) \sqcap \mathcal{D}(Q)
\end{aligned}$$

Executing the program: **gfp** $\mathcal{D}(P)$.

Sequential Computation = Parallel Computation

We obtain the same result if we execute P in parallel or if we replace all parallel $\|$ by a sequential operator ; (a transformation we write $\mathtt{seq}\ P$) defined as follows:

$$\mathcal{D}(P\ ;\ Q) \triangleq \mathcal{D}(Q) \circ \mathcal{D}(P)$$

Let **fix** f be the set of fixpoints of a function f.

Theorem (Equivalence Between Sequential and Parallel Operators)

$\textbf{fix}\ \mathcal{D}(\mathtt{seq}\ P) = \textbf{fix}\ \mathcal{D}(P)$

Very weak requirements on the parallel hardware

The semantics make the following assumptions on the memory consistency model and cache coherency protocol:

(ATOM) Load and store instructions must be atomic.

(EC) The caches must eventually become coherent.

(OTA) Values cannot appear *out-of-thin-air*.

Parallel Propagation

Memory:

$x = [1..10]$
$y = [1..10]$
$z = [1..10]$

Propagators:

$propagate(d, x + 3 \leq y) =$
  update $d(x)_u$ to $d(y)_u - 3$
  update $d(y)_\ell$ to $d(x)_\ell + 3$

$propagate(d, x + 6 \leq z) =$
  update $d(x)_u$ to $d(z)_u - 6$
  update $d(z)_\ell$ to $d(x)_\ell + 6$

$$propagate(d, x + 3 \leq y) \,||\, propagate(d, x + 6 \leq z)$$

Issues of Parallel Propagation

Memory:

$x = [1..7]$
$y = [1..10]$
$z = [1..10]$

Propagators:

$$\begin{array}{ll} & \text{update } d(x)_u \text{ to } d(y)_u - 3 \\ \| & \text{update } d(y)_\ell \text{ to } d(x)_\ell + 3 \\ \| & \text{update } d(x)_u \text{ to } d(z)_u - 6 \\ \| & \text{update } d(z)_\ell \text{ to } d(x)_\ell + 6 \end{array}$$

Issue 1: data race? Parallel update of the same integer $d(x)_u$.

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\end{array}
$$

Issue 1: data race? Parallel update of the same integer $d(x)_u$.
$\Rightarrow$ **Solution**: Use atomic load and store!
$\Rightarrow$ **In CUDA**: Integer load and store are atomic by default!

Issue 2: nondeterminism? $d(x)_u$ can be equal to 4 or 7 depending on the order of execution.

Solution: Use a fixpoint loop

While something is changing we reexecute all the propagators!

Memory:

$x = [1..4]$
$y = [1..10]$
$z = [1..10]$

Propagators:

```
fp (
        update $d(x)_u$ to $d(y)_u - 3$
     || update $d(y)_\ell$ to $d(x)_\ell + 3$
     || update $d(x)_u$ to $d(z)_u - 6$
     || update $d(z)_\ell$ to $d(x)_\ell + 6$   )
```

Solution: Use a fixpoint loop

While something is changing we reexecute all the propagators!

Memory:

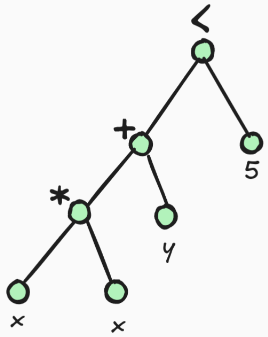
$x = [1..4]$
$y = [1..10]$
$z = [1..10]$

Need to add a condition for progress!

update $d(x)_\ell$ to v iff $v > d(x)_\ell$
update $d(x)_u$ to v iff $v < d(x)_u$

Towards an Efficient GPU Constraint Solver

Representation of Propagators



- Represented using `shared_ptr` and `variant` data structures.

  ⇒ Uncoalesced memory accesses.

- Code similar to an interpreter:

```
switch(term.index()) {
  case IVar:
  case INeg:
  case IAdd:
  case IMul:
  // ...
```

  ⇒ Thread divergence.

Ternary Normal Form (TNF)

We simplify the representation of constraints to ternary constraints of the form:

- `x = y <op> z` where `x,y,z` are variables.
- The operators are $\{+, -, /, *, \%, \mathit{min}, \mathit{max}, \leq, =\}$.

Example

The constraint $x + y \neq 2$ is represented by:

```
t1 = x + y
ZERO = (t1 = TWO)
```

where `ZERO` and `TWO` are two variables with constant values.

Bytecode Representation

The ternary form of a propagator holds on 16 bytes:

```
struct bytecode_type {
  Sig op;
  AVar x;
  AVar y;
  AVar z;
};
```

- **Uniform representation** of propagators in memory $\Rightarrow$ coalesced memory accesses.
- Limited number of operators + sorting $\Rightarrow$ reduced thread divergence.

Problems in Ternary Normal Form

Drawback 1 of TNF: increase in number of propagators and variables.

Problem	**#Variables**	**#Constraints**
team-assignment	35445 (x2.2)	45197 (x1.8)
generalized-peacable-queens	20186 (x6.9)	25519 (x3.1)
spot5	22053 (x19.8)	29065 (x3.6)
accap	2319 (x5.2)	2782 (x3)
wordpress	92695 (x139)	122921 (x4)

Observation: The increase in constraints is relatively stable across problems (between 1x and 6x), but not the increase in variables (between 1.5x and 139x).

Benchmarks

Average gain of TNF/bytecode representation instead of tree-based propagators (on 16 MiniZinc instances):

- **Fixpoint iterations per second**: +553% (845k)
- $\Rightarrow$ Huge gain: x5 more iterations.

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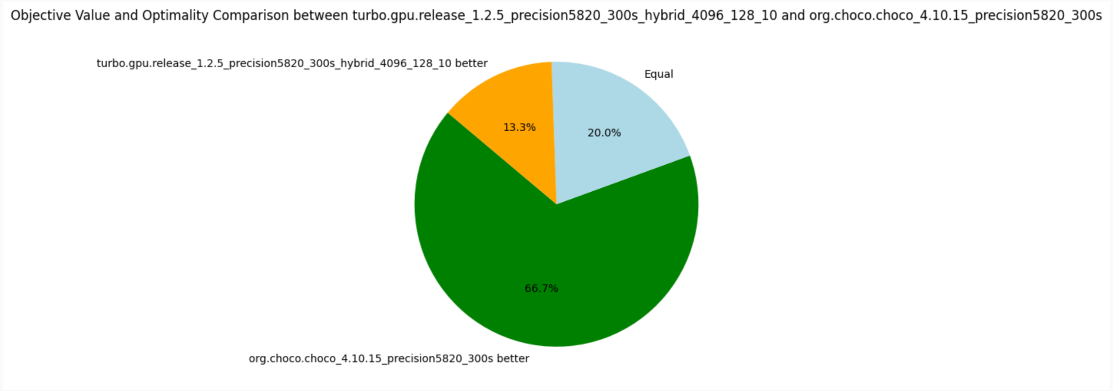
- ⇒ More modest increase in nodes per second. Why?
- **Fixpoint iterations per node**: +138% (40 instead of 17 before).
- ⇒ Slower convergence of the fixpoint loop.
- **Memory footprint of propagators**: -90% (0.85MB instead of 7MB).
- **Memory footprint of variables**: +718% (402.75KB instead of 4.98KB).

Benchmarks: Turbo vs Choco v4.15

Comparison of the best objective values found.



What’s Next?

In-Progress: Parallel Consistency Algorithm

- **Goal**: Avoid executing all propagators in each iteration of the fixpoint loop.
- Constraint network as a sparse graph.
- Waking up propagators using GPU-based algorithms (SpMV and scan).

In-Progress: Table Abstract Domain

- **Context:** In constraint programming, *global constraints* are propagators with dedicated inference algorithms for subproblems, e.g., `alldifferent(`$[x_1,\ldots,x_n]$`)`.
- Research question: Which global constraints can be generalized into abstract domains?

Collaboration with Éric Monfroy

We are working on the *Table abstract domain* generalizing the well-known `table` constraint:

$$\begin{aligned} &(x \geq 4 \land y > 1 \land z < 3)\\ \lor&(x = 1 \land y = 2 \land z = 3)\\ \lor&(x > 1 \land y > 1 \land z > 3)\end{aligned}$$

Perspective: Towards automatic creation of the abstract domain

Research question: Given a set of abstract domains and reduced products, how to build the most efficient one to solve a given formula?

$$\begin{tikzcd}
 & \varphi \arrow[dl, "\llbracket\rrbracket"'] \arrow[dr, "\llbracket\rrbracket^\flat"] & \\
A_1^\sharp \times \ldots \times A_n^\sharp \arrow[rr, shift left, "\gamma"] & & C^\flat \arrow[ll, shift left, "\alpha"]
\end{tikzcd}$$

- How to create an appropriate combination of abstract domains for a particular formula?
- "Type inference": In which abstract domain goes each subformula $\varphi_i \in \varphi$?

Conclusion

C++ Abstraction: Lattice Land Project

`lattice-land` is a collection of libraries abstracting our parallel model.

It provides various data types and fixpoint engine:

- `ZLB`, `ZUB`: increasing/decreasing integers.
- `B`: Boolean lattices.
- `VStore`: Array (of lattice elements).
- `IPC`: Arithmetic constraints.
- `GaussSeidelIteration`: Sequential CPU fixed point loop.
- `AsynchronousIteration`: GPU-accelerated fixed point loop.
- ...

```
void max(int tid, const int* data, ZLB& m) {
  m.tell(data[tid]);
}
AsynchronousIteration::fixpoint(max);
```

https://github.com/lattice-land

Conclusion

Data races occur rarely, so we should avoid working so much to avoid them.

Properties of the model

A Variant of Concurrent Constraint Programming on GPU (AAAI 2022)[13].

- Correct: Proofs that $P; Q \equiv P||Q$, parallel and sequential versions produce the same results.
- Restartable: Stop the program at any time, and restart on partial data.
- Modular: Add more threads without fear of breaking existing code.
- Weak memory consistency: Very few requirements on the underlying memory model $\Rightarrow$ wide compatibility across hardware, unlock optimization.

[13] `https://ptal.github.io/papers/aaai2022.pdf`