Abstract Satisfaction

Lattice Theory for Parallel Programming

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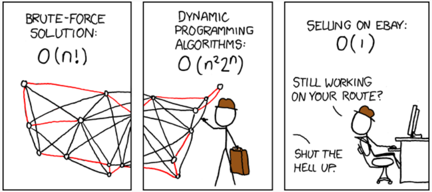
This lecture in a nutshell!

We present the "fusion" of...

Constraint reasoning + Abstract interpretation

(and lattice theory)



that gives us abstract satisfaction.

This lecture in a nutshell!

We present the “fusion” of...

Constraint reasoning + Abstract interpretation

WHY?

- Combining constraint solvers.
- Constructing sound propagators over complex domains.
- Constraint solving on GPUs.

that gives us abstract satisfaction.

On the Menu

- **Abstract Satisfaction** (connection between logic and constraint reasoning)
- **Abstract Constraint Programming** (expressive reasoning framework)
- **Abstract Constraint Programming on GPU** (efficient reasoning framework)

Abstract Satisfaction

Syntax of First-Order Logic (FOL)

Let $S = \langle X, F, P\rangle$ be a *first-order signature* where X set of variables, F set of function symbols and P set of predicate symbols.

$$
\begin{array}{rcll}
\langle t\rangle & ::= & x & \textit{variable } x \in X \\
& | & f(t, \dots, t) & \textit{function } f \in F
\end{array}
$$

$$
\begin{array}{rcll}
\langle \varphi\rangle & ::= & p(t, \dots, t) & \textit{predicate } p \in P \\
& | & \neg\varphi & \textit{negation} \\
& | & \varphi \diamond \varphi & \textit{connector } \diamond \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\} \\
& | & \exists x,\ \varphi & \textit{existential quantifier} \\
& | & \forall x,\ \varphi & \textit{universal quantifier}
\end{array}
$$

Let Φ the set of well-formed formulas.

Semantics of FOL

A *structure* A is a tuple $(\mathbb{U}, [\![]\!]_F, [\![]\!]_P)$ where

1. $\mathbb{U}$ is a non-empty set of elements—called the *universe of discourse*,
2. $[\![]\!]_F$ is a function mapping function symbols $f \in F$ with arity n to interpreted functions $[\![f]\!]_F : \mathbb{U}^n \to \mathbb{U}$, and
3. $[\![]\!]_P$ is a function mapping predicate symbols $p \in P$ with arity n to interpreted predicates $[\![p]\!]_P \subseteq \mathbb{U}^n$.

An *assignment* is a function $X \to \mathbb{U}$ mapping variables to values. We denote the set of assignment by **Asn**. Let $\rho \in$ **Asn**, we write $\rho[x \mapsto d]$ the assignment in which we updated the value of x by d in ρ.

Entailment

The syntax and semantics are related by the ternary relation $A \vDash_\rho \varphi$, called the *entailment*, where A is a structure, $\rho \in \textbf{Asn}$ and $\varphi \in \Phi$. It is read as "the formula φ is satisfied by the assignment ρ in the structure A". We first give the interpretation function $[\![]\!]_\rho$ for evaluating the terms of the language:

$$\begin{aligned}
[\![x]\!]_\rho &= \rho(x) \text{ if } x \in X \\
[\![f(t_1, \dots, t_n)]\!]_\rho &= [\![f]\!]_F([\![t_1]\!]_\rho, \dots, [\![t_n]\!]_\rho)
\end{aligned}$$

The relation $\vDash$ is defined inductively as follows:

$$\begin{array}{ll}
A \vDash_\rho p(t_1, \dots, t_n) & \text{iff } ([\![t_1]\!]_\rho, \dots, [\![t_n]\!]_\rho) \in [\![p]\!]_P \\
A \vDash_\rho \varphi_1 \wedge \varphi_2 & \text{iff } A \vDash_\rho \varphi_1 \text{ and } A \vDash_\rho \varphi_2 \\
A \vDash_\rho \varphi_1 \vee \varphi_2 & \text{iff } A \vDash_\rho \varphi_1 \text{ or } A \vDash_\rho \varphi_2 \\
A \vDash_\rho \neg\varphi & \text{iff } A \vDash_\rho \varphi \text{ does not hold} \\
A \vDash_\rho \exists x,\ \varphi & \text{iff there exists } d \in \mathbb{U} \text{ such that } A \vDash_{\rho[x \mapsto d]} \varphi \\
A \vDash_\rho \forall x,\ \varphi & \text{iff for all } d \in \mathbb{U},\ \text{we have } A \vDash_{\rho[x \mapsto d]} \varphi
\end{array}$$

Concrete Domain

Given a structure A, we define the *concrete interpretation function* as:

$$\llbracket . \rrbracket^\flat : \Phi \to \mathcal{P}(\textbf{Asn})$$
$$\llbracket \varphi \rrbracket^\flat = \{\rho \in \textbf{Asn} \mid A \vDash_\rho \varphi\}$$

- We call the *concrete domain* the lattice $D^\flat \triangleq \langle \mathcal{P}(\textbf{Asn}), \subseteq \rangle$ with $\llbracket . \rrbracket^\flat$.
- A *solution* of the formula φ is an assignment $s \in \llbracket \varphi \rrbracket^\flat$.
- **Example** in the theory of standard integer arithmetics (and $X = \{x, y\}$):

$$
\begin{aligned}
&\llbracket x < y \wedge x \geq 0 \rrbracket^\flat = \{ \\
&\quad \{x \mapsto 0, y \mapsto 1\} \\
&\quad \{x \mapsto 0, y \mapsto 2\} \\
&\ldots \\
&\quad \{x \mapsto 1, y \mapsto 2\} \\
&\ldots \\
&\}
\end{aligned}
$$

One Problem, Many Communities, Many Formalisms

Many communities emerged to solve the same problem: find ρ such that $A \vDash_\rho \varphi$.

BUT they (generally) focus on different fragments of FOL:

- Propositional fragment (SAT): $(a \vee b) \wedge (\neg b \vee c)$ with $a, b, c \in \{0, 1\}$.
- Pseudo-Boolean fragment: $\sum_{1 \leq i \leq n} c_i * a_i \leq c_0$ with $a_i \in \{0, 1\}$ and c_i some integers constants.
- Linear programming (LP): $\sum_{1 \leq i \leq n} c_i * b_i \leq b_0$ with $b_i \in \mathbb{R}$ and c_i some real constants.
- Integer linear programming (ILP): $\sum_{1 \leq i \leq n} c_i * b_i \leq b_0$ with $b_i \in \mathbb{Z}$ and c_i some integer constants.
- Mixed integer linear programming (MILP): $\sum_{1 \leq i \leq n} c_i * b_i \leq b_0$ with $b_i \in \mathbb{Z}$ or $b_i \in \mathbb{R}$ and c_i some integer or real constants.
- Uninterpreted fragment (logic programming).
- Discrete constraint programming: $\langle X, D, C \rangle$ with $D_i \in \mathcal{P}_f(\mathbb{Z})$.
- Continuous constraint programming: $\langle X, D, C \rangle$ with $D_i \in \mathcal{I}(\mathbb{R})$.
- Satisfiability modulo theories (SMT).
- ...

One Theory to Rule Them All?

- SAT [DHK13]
- SMT [DHK14]
- Logic programming [Cou20]
- Constraint programming ($\mathbb{R}$) [Pel+13]
- Constraint programming ($\mathbb{Z}$) [Tal+19]
- Linear programming [CH78]
- Answer set programming
- ...

$\rightarrow$ Abstract domains

What is an *abstract domain*?

It is a lattice with some operations.

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A tuple $\langle S, \sqsubseteq, \sqcup, \sqcap, \bot, \top \rangle$ where S is a set.

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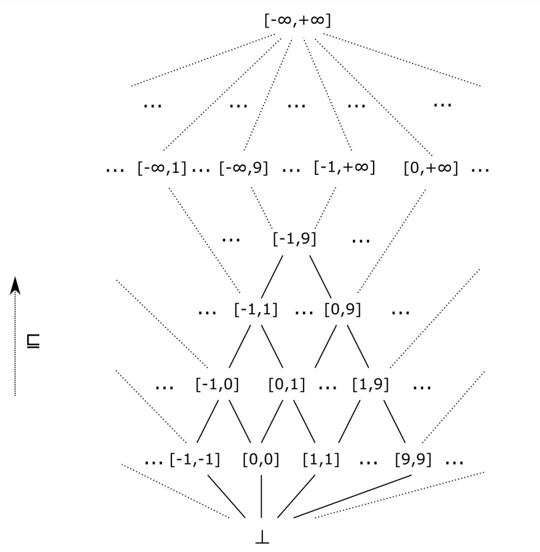
Example: Interval Lattice

- $S \triangleq \{[a, b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{\infty\}, a \leq b\} \cup \{\bot\}$
- $[a, b] \sqsubseteq [c, d] \Leftrightarrow a \geq c \wedge b \leq d$
- $\top \triangleq [-\infty, \infty]$
- $[a, b] \sqcap [c, d] \triangleq [\max\{a, c\}, \min\{b, d\}]$



Simple Logic of Intervals

- **Logic**: $\Phi \triangleq x \leq k \mid x \geq k \mid \Phi \wedge \Phi \mid \Phi \vee \Phi$. (only 1 variable)

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  - $[\![x \leq k]\!] \triangleq [-\infty, k]$
  - $[\![x \geq k]\!] \triangleq [k, \infty]$
  - $[\![\varphi \wedge \varphi']\!] \triangleq [\![\varphi]\!] \sqcap [\![\varphi']\!]$
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- **Example**:
  - $[\![(x \leq 10 \wedge x \geq 0) \vee (x \geq 5)]\!]$
  - $[\![x \leq 10 \wedge x \geq 0]\!] \sqcup [\![x \geq 5]\!]$
  - $([\![x \leq 10]\!] \sqcap [\![x \geq 0]\!]) \sqcup [\![x \geq 5]\!]$
  - $([-\infty, 10] \sqcap [0, \infty]) \sqcup [5, \infty]$
  - $[0, 10] \sqcup [5, \infty]$
  - $[0, \infty]$

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  - $[\![x \leq 10 \wedge x \geq 0]\!] \sqcup [\![x \geq 5]\!]$
  - $([\![x \leq 10]\!] \sqcap [\![x \geq 0]\!]) \sqcup [\![x \geq 5]\!]$
  - $([-\infty, 10] \sqcap [0, \infty]) \sqcup [5, \infty]$
  - $[0, 10] \sqcup [5, \infty]$
  - $[0, \infty]$
- **Soundness**: $[\![\varphi]\!]^\flat \subseteq [\![\varphi]\!]$ (compute all solutions).
- **Completeness**: $[\![\varphi]\!]^\flat \supseteq [\![\varphi]\!]$ (compute only solutions).
  Intervals are not complete: $[\![x \leq 10 \vee x \geq 15]\!] = [-\infty, \infty]$ (intervals cannot represent "holes").

What About Multiple Variables?

We lift interval to a function $X \to \mathit{Itv}$ mapping variables to intervals where Itv is the interval lattice.

Now, we can define (with $x \in X$ any variable):

- $[\![x \leq k]\!] \triangleq \{x \mapsto [-\infty, k]\}$.
- $[\![x \geq k]\!] \triangleq \{x \mapsto [k, \infty]\}$.
- ...

Example: $[\![x \leq 0 \wedge y \geq 0]\!] = \{x \mapsto [-\infty, 0], y \mapsto [0, \infty]\}$.

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How to compute solutions of more expressive logic?

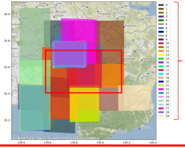
Abstract Constraint Programming

Constraint Programming

Constraint programming: FOL without quantifiers, $\mathbb{U} = \mathbb{Z}$ and arithmetic constraints.

- **Declarative paradigm**: specify your problem and let the computer solves it for you.
- **Many applications**: scheduling, bin-packing, hardware design, satellite imaging, ...
- **Constraint programming** is one approach to solve such combinatorial problems.
- Other approaches include SAT, linear programming, SMT, MILP, ASP,...

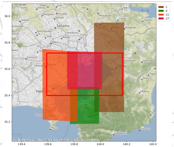
Satellite image mosaic



After a query with certain parameters:

N = 30 satellite images

Find the cover with the minimum number of images (NP-Hard)

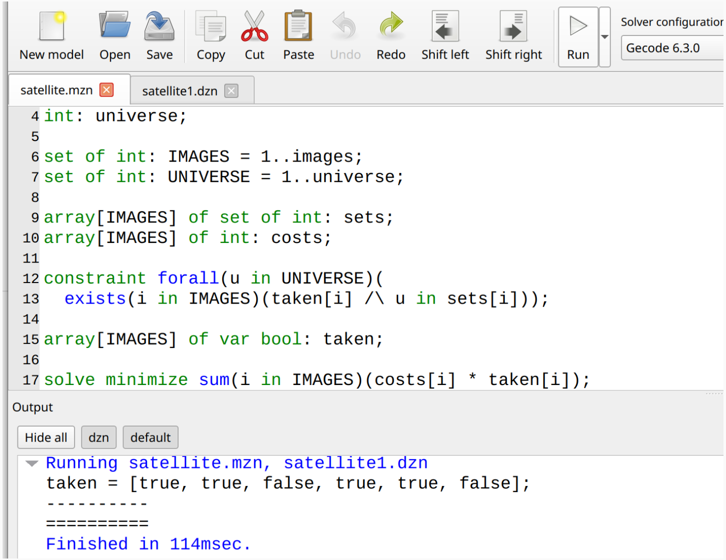


Build the mosaic

1

[1] Combarro et al., Constraint Model for the Satellite Image Mosaic Selection Problem, CP 2023

Constraint model of satellite imaging in MiniZinc:



Constraint Network

Constraint Network

Let X be a finite set of variables and C be a finite set of constraints.
A *constraint network* is a pair $P = \langle d, C \rangle$ such that $d \in X \rightarrow Itv$ is the *domain* of the variables where Itv is the set of intervals.

Note: It is just a "format" to represent quantifier-free logical formulas where variables have bounded domains.

Example

$$\langle \{x \mapsto [0,2], y \mapsto [2,3]\}, \{x \leq y - 1\} \rangle$$

A solution is $\{x \mapsto 0, y \mapsto 2\}$.

Constraints with Multiple Variables

- We already have: $[\![x \leq k]\!] \triangleq \{x \mapsto [-\infty, k]\}$.
- Also: $[\![x = k]\!] \triangleq \{x \mapsto [k, k]\}$.

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Solution: give more information to the interpretation function.

- $\mathcal{I}[\![.]\!] \in \Phi \times (X \to Itv) \to (X \to Itv)$
- $\mathcal{I}[\![x = y]\!]d \triangleq \{x \mapsto d(x) \sqcap d(y), y \mapsto d(x) \sqcap d(y)\}$

Example: Let $d = \{x \mapsto [0, 5], y \mapsto [5, 10]\}$, then $\mathcal{I}[\![x = y]\!]d = \{x \mapsto [5, 5], y \mapsto [5, 5]\}$.

How to Deal with Conjunction?

- Before, we had $[\![\varphi \wedge \varphi']\!] \triangleq [\![\varphi]\!] \sqcap [\![\varphi']\!]$.
- Now, we can lift this function to

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- Instead, we can use functional composition:

$$\mathcal{I}[\![\varphi \wedge \varphi']\!]d \triangleq (\mathcal{I}[\![\varphi]\!] \circ \mathcal{I}[\![\varphi']\!])d$$

Computing Solutions of Constraint Network

A constraint network $\langle d, C\rangle$ is a conjunctive collection of constraints. So we can compute the set of solutions using:

$$\mathcal{I}[\![c_1 \wedge c_2 \wedge \ldots \wedge c_n]\!] = \mathcal{I}[\![c_1]\!] \circ \mathcal{I}[\![c_2]\!] \circ \ldots \circ \mathcal{I}[\![c_n]\!]$$

Example: Let $\langle d, \{x = y, y = z\}\rangle$ be a constraint network with $d = \{x \mapsto [2,2], y \mapsto [1,2], z \mapsto [0,2]\}$, then:

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= & \mathcal{I}[\![x = y]\!](\mathcal{I}[\![y = z]\!](d)) \\
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We are not very precise... $z = [1,2]$ instead of $z = [2,2]$.

Computing the Greatest Fixpoint

- $d_1 = \{x \mapsto [2,2], y \mapsto [1,2], z \mapsto [0,2]\}$.
- $d_2 = \mathcal{I}[\![x = y \wedge y = z]\!]d_1 = \{x \mapsto [2,2], y \mapsto [2,2], z \mapsto [1,2]\}$.

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For all formulas φ, $\mathcal{I}[\![\varphi]\!]$ is a *monotone function*.
Hence, we are guaranteed to find the greatest fixpoint, which is unique (Tarski theorem).

Constraint propagation is an approach to compute efficiently the *greatest fixpoint*:

$$\mathbf{gfp}_d\ \mathcal{I}[\![c_1]\!] \circ \ldots \circ \mathcal{I}[\![c_n]\!]$$

Propagate and Search

The main algorithm behind discrete constraint solvers:

function $\textsc{solve}(d, \{c_1, \ldots, c_n\})$
  $d \leftarrow \mathbf{gfp}_d\ \mathcal{I}[\![c_1]\!] \circ \ldots \circ \mathcal{I}[\![c_n]\!]$
  if $\forall x \in X, \exists v \in \mathbb{Z},\ d(x) = [v, v]$ **then return** $\{d\}$
  else if $\exists x \in X,\ d(x) = \bot$ **then return** $\{\}$
  else
    $\langle d_1, \ldots, d_n \rangle \leftarrow \texttt{split}(d)$
    return $\bigcup_{i=0}^{n} \texttt{solve}(d_i, C)$
  end if
end function

Thanks to the split function, the algorithm is **sound and complete**.

Soundness

Closure Operator

The concrete interpretation function $[\![.]\!]^\flat$ can be lifted to a closure operator over the concrete domain defined as follows ($D^\flat \triangleq \langle \mathcal{P}(X \to \mathbb{U}), \subseteq \rangle$):

$$\mathcal{F}[\![.]\!] : \Phi \to (D^\flat \to D^\flat)$$
$$\mathcal{F}[\![\varphi]\!]A \triangleq A \cap [\![\varphi]\!]^\flat$$

Theorem

For all formulas $\varphi \in \Phi$, $\mathcal{F}[\![\varphi]\!]$ is a closure operator over $D^\flat$.

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Theorem

For all formulas $\varphi \in \Phi$, $\mathcal{F}[\![\varphi]\!]$ is a closure operator over $D^\flat$.

It is helpful to construct $\mathcal{F}[\![.]\!]$ inductively (easier to prove an abstraction is sound):

$$\begin{aligned}
\mathcal{F}[\![\mathit{true}]\!]A &= A \\
\mathcal{F}[\![\mathit{false}]\!]A &= \{\} \\
\mathcal{F}[\![p(t_1, \ldots, t_n)]\!]A &= \{\rho \in A \mid ([\![t_1]\!]\rho, \ldots, [\![t_n]\!]\rho) \in [\![p]\!]_P\} \\
\mathcal{F}[\![\neg\varphi]\!]A &= A \setminus \mathcal{F}[\![\varphi]\!]\mathbf{Asn} \\
\mathcal{F}[\![\varphi_1 \wedge \varphi_2]\!]A &= \mathcal{F}[\![\varphi_1]\!]A \cap \mathcal{F}[\![\varphi_2]\!]A \\
\mathcal{F}[\![\varphi_1 \vee \varphi_2]\!]A &= \mathcal{F}[\![\varphi_1]\!]A \cup \mathcal{F}[\![\varphi_2]\!]A
\end{aligned}$$

Solutions of a FOL Formula

The solutions of φ are given by the greatest fixed point $\mathit{gfp}^{\subseteq}\ \mathcal{F}[\![\varphi]\!]$.

Lemma

$\mathit{gfp}^{\subseteq}\ \mathcal{F}[\![\varphi]\!] = [\![\varphi]\!]^{\flat}$

Abstract Domain

Definition

An abstract domain (for constraint reasoning) is a bounded lattice $\langle A^\sharp, \sqsubseteq, \sqcup, \sqcap, \bot, \top, \mathcal{F}^\sharp[\![.]\!]\rangle$ such that:

- Every element of $A^\sharp$ is representable in a machine.
- The operations on $A^\sharp$ are efficiently computable.
- $\mathcal{F}^\sharp[\![.]\!] : \Phi \to (A^\sharp \to A^\sharp)$ is a sound abstraction of $\mathcal{F}[\![.]\!]$.

The concrete and abstract semantics are connected by a Galois connection:

$$\langle \mathcal{P}(X \to \mathbb{U}), \subseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A^\sharp, \sqsubseteq\rangle$$

Soundness: The abstract function $\mathcal{F}^\sharp[\![\varphi]\!]$ should not remove any solution!

Galois Connection

Let $\langle C, \subseteq \rangle$ and $\langle A, \sqsubseteq \rangle$ be lattices, and $\alpha : C \to A$ and $\gamma : A \to C$ two maps.

Definition (Galois Connection)

The pair (α, γ) is a Galois connection iff, $\forall c \in C, \forall a \in A,\ \alpha(c) \sqsubseteq a \Leftrightarrow c \subseteq \gamma(a)$. We denote this Galois connection as:

$$\langle C, \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A, \sqsubseteq \rangle$$

Definition (Alternative Characterization of Galois Connection)

The pair (α, γ) is a Galois connection iff $\forall c \in C,\ \forall a \in A$,

- (Gal1) $c \leq \gamma(\alpha(c))$ and $\alpha(\gamma(a)) \leq a$.
- (Gal2) α and γ are order-preserving.

Exercise: Prove the equivalence between both definitions.

Soundness

The concrete and abstract semantics are connected by a Galois connection ($D^\flat = \mathcal{P}(X \to \mathbb{U})$):

$$\langle D^\flat, \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A^\sharp, \sqsubseteq \rangle$$

- **What we have:** A concrete property $S \in D^\flat$ and $\mathcal{F}[\![\varphi]\!] : D^\flat \to D^\flat$ a closure operator filtering from S the assignments that are not solutions from φ.
- **What we want:** An abstract function $\mathcal{F}^\sharp[\![\varphi]\!] : A^\sharp \to A^\sharp$, which computes an over-approximation (aka. a superset) of the solutions of φ.
- We are going to define the notion of **soundness**.

Soundness

$$\langle D^\flat, \subseteq\rangle \xleftarrow[\alpha]{\gamma} \langle A^\sharp, \sqsubseteq\rangle$$

Defining "$\mathcal{F}^\sharp[\![\varphi]\!]$ over-approximates $\mathcal{F}[\![\varphi]\!]$":

- By (Gal1), we have $c \subseteq \gamma(\alpha(c))$.

Soundness

$$\langle D^\flat, \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle A^\sharp, \sqsubseteq \rangle$$

Defining "$\mathcal{F}^\sharp[\![\varphi]\!]$ over-approximates $\mathcal{F}[\![\varphi]\!]$":

- By (Gal1), we have $c \subseteq \gamma(\alpha(c))$.
- Since $\mathcal{F}[\![\varphi]\!]$ is reductive ($\mathcal{F}[\![\varphi]\!](c) \subseteq c$) we also have $\mathcal{F}[\![\varphi]\!](c) \subseteq \gamma(\alpha(c))$.

Soundness

$$\langle D^\flat, \subseteq \rangle \xleftarrow[\alpha]{\gamma} \xrightarrow{} \langle A^\sharp, \sqsubseteq \rangle$$

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- Equivalently and for clarity, we can write $\mathcal{F}[\![\varphi]\!] \mathrel{\dot\subseteq} \gamma \circ \alpha$ where $\dot\subseteq$ is the pointwise order.

Soundness

$$\langle D^\flat, \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A^\sharp, \sqsubseteq \rangle$$

Defining "$\mathcal{F}^\sharp[\![\varphi]\!]$ over-approximates $\mathcal{F}[\![\varphi]\!]$":

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- Once we are in the abstract, we wish to compute using $\mathcal{F}^\sharp[\![\varphi]\!]$, and therefore, we must have:

$$\mathcal{F}[\![\varphi]\!] \mathrel{\dot{\subseteq}} \gamma \circ \mathcal{F}^\sharp[\![\varphi]\!] \circ \alpha$$

Soundness

$$\langle D^\flat, \subseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A^\sharp, \sqsubseteq\rangle$$

Defining "$\mathcal{F}^\sharp[\![\varphi]\!]$ over-approximates $\mathcal{F}[\![\varphi]\!]$":

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Lemma

$\mathcal{F}^\sharp[\![\varphi]\!]a \triangleq a$ is a sound overapproximation of $\mathcal{F}[\![\varphi]\!]$.

Interval Abstract Domain

The abstract domain of integer intervals is
$\mathcal{I}^\sharp \triangleq \langle X \to \mathcal{I}, \dot{\sqsubseteq}, \dot{\sqcup}, \dot{\sqcap}, x \in X \mapsto \bot, x \in X \mapsto [-\infty, \infty], \mathcal{I}[\![.]\!]\rangle$ where $\dot{\sqsubseteq}, \dot{\sqcup}, \dot{\sqcap}$ are pointwise interval operations.

We have the Galois connection:

$$\langle X \to \mathcal{P}(\mathbb{Z}), \dot{\subseteq}\rangle \underset{\overline{\alpha}}{\overset{\overline{\gamma}}{\leftrightarrows}} \langle X \to \mathcal{I}, \dot{\sqsubseteq}\rangle$$
$$\overline{\alpha}(S) \triangleq x \in X \mapsto [min\ S(x), max\ S(x)]$$
$$\overline{\gamma}(R) \triangleq x \in X \mapsto \{c \in \mathbb{Z} \mid \lfloor R(x) \rfloor \leq c \leq \lceil R(x) \rceil\}$$

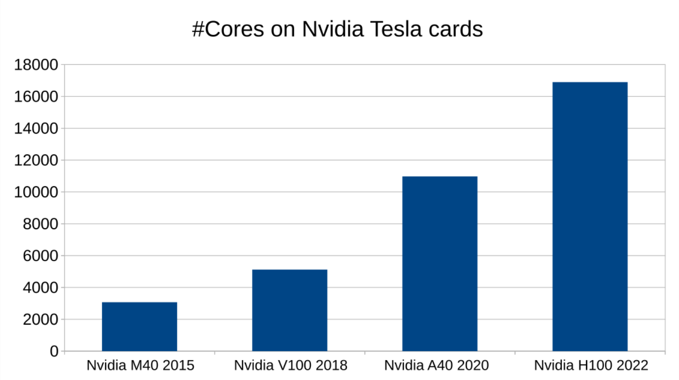
Exercise: Prove the soundness of $\mathcal{I}[\![x = y]\!]$.

Constraint Programming on GPU

Why Constraint Programming on GPU?

Why CP on GPU?

CPU clock speed is stagnating, GPU #cores is increasing quickly each year.



Easy speed-up: same code but faster.

Why CP on GPU?

- Machine learning (deep learning, reinforcement learning, . . .) has seen tremendous speed-ups (e.g. 100x, 1000x) by using GPU.
- Some (sequential) optimizations on CPU are made irrelevant if we can explore huge state space faster.

Can we replicate the success of GPU on machine learning applications to combinatorial optimization?

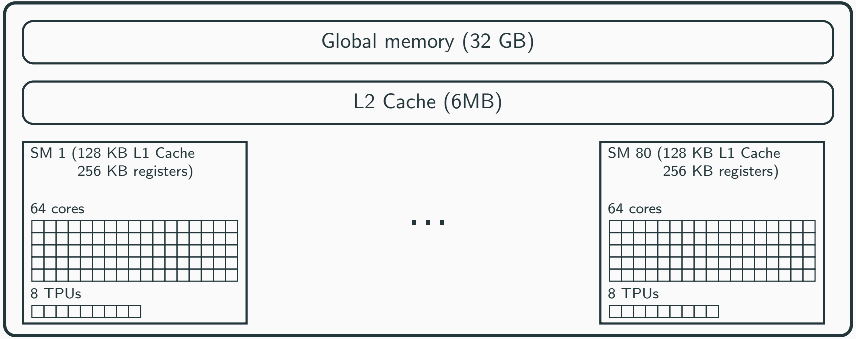
Constraint Programming on GPU

The rest of this talk:

- GPU Architecture.
- Challenges of Constraint Programming on GPU.
- Parallel Model of Computation.
- Ternary Constraint Network.

GPU Architecture

(Simplified) Architecture of the GPU Nvidia V100



5120 cores on a single V100 GPU @ 1290MHz

Whitepaper: `https://images.nvidia.com/content/volta-architecture/pdf/volta-architecture-whitepaper.pdf`

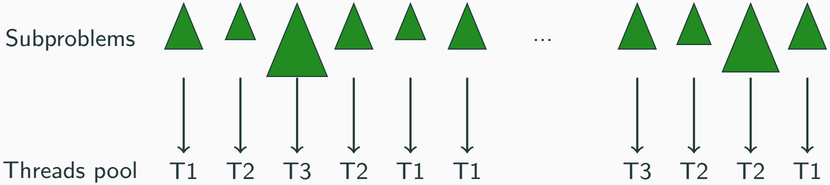
Programming Challenges

- **Memory coalescence**: the way to access the data is important (factor 10).
- **Thread divergence**: each thread within a warp (group of 32 threads) should execute the same instructions.
- **Memory allocation** (dynamic data structures): costly on GPU, everything is generally pre-allocated.
- **Other limitations**: small cache, limited number of lines of code, limited STL...

Challenges of Constraint Programming on GPU

On CPU: Embarrasingly Parallel Search (EPS)[2]

- **Idea**: Divide the problem into many subproblems beforehand (e.g. $N \times 30$ with N the number of threads).
- **Intuition**: Statistically, there is little chance a subproblem takes longer than the sum of the other subproblems.



[2] A. Malapert et al., 'Embarrassingly Parallel Search in Constraint Programming', JAIR, 2016

On CPU: Embarrasingly Parallel Search (EPS)

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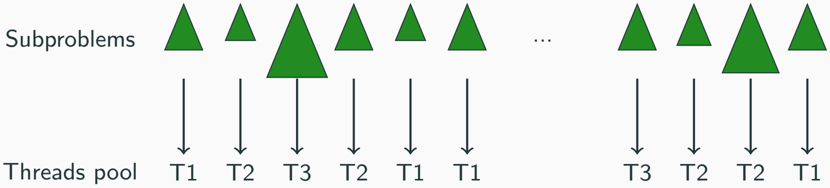

⇒ **Other approach:** In modern solvers (e.g., Choco, OR-Tools), they use a portfolio approach (e.g., different *split* strategy on the *same problem*).

On CPU: Embarrasingly Parallel Search (EPS)

- **Idea**: Divide the problem into many subproblems beforehand (e.g. $N \times 30$ with N the number of threads).
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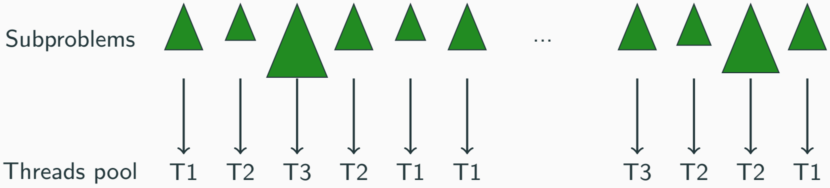


On GPU architectures, 1 subproblem per thread is not efficient (limited cache).

$\Rightarrow$ Need to parallelize propagation: $\mathbf{gfp}_d\ \mathcal{I}[\![c_1]\!] \circ \ldots \circ \mathcal{I}[\![c_n]\!]$.

Where is the Challenge?

Parallelizing $\mathbf{gfp}_d\ \mathcal{I}[\![c_1]\!] \circ \ldots \circ \mathcal{I}[\![c_n]\!]$ is challenging because constraints share variables, and we have typical *shared state memory* issues such data races and inefficiencies.

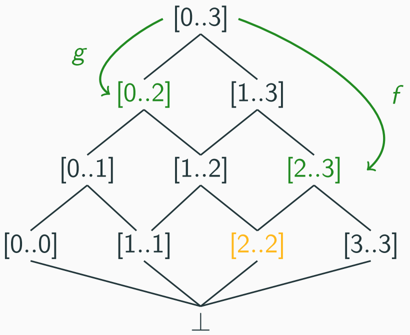
Contributions

- **New parallel model of computation** to execute propagators in parallel[2]:
  $\mathbf{gfp}_d\ \mathcal{I}[\![c_1]\!] \parallel \ldots \parallel \mathcal{I}[\![c_n]\!]$
- **Ternary constraint network**: representation of constraints suited for GPU architectures[3].
- **First general constraint solver fully executing on GPU.**
  $\Rightarrow$ **Open-source**: Publicly available on `https://github.com/ptal/turbo`.

[2] P. Talbot et al., *A Variant of Concurrent Constraint Programming on GPU*, AAAI, 2022.
[3] P. Talbot, *A GPU-based Constraint Programming Solver*, AAAI, 2026.

Parallel Model of Computation

Parallel Model of Computation



- $f(x) \triangleq x \sqcap [2..\infty]$ models the constraint $x \geq 2$.
- $g(x) \triangleq x \sqcap [-\infty..2]$ models the constraint $x \leq 2$.
- Parallel execution: `f || g = [2..2]`

Example of Parallel Propagation

Let's consider $\mathcal{I}[\![x \leq 4 \land x \leq 5]\!] = \mathcal{I}[\![x \leq 4]\!] \parallel \mathcal{I}[\![x \leq 5]\!]$

Memory:

$$\boxed{x = [-\infty, \infty]}$$

Propagators:

$$\begin{array}{lll} & x \leftarrow [-\infty, 4] & (\mathcal{I}[\![x \leq 4]\!]) \\ \parallel & x \leftarrow [-\infty, 5] & (\mathcal{I}[\![x \leq 5]\!]) \end{array}$$

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$$\boxed{x = [-\infty, ?]}$$

Propagators:

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Issue 1: data race? Parallel update of the same variable: upper bound of x.

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Let's consider $\mathcal{I}[\![x \leq 4 \land x \leq 5]\!] = \mathcal{I}[\![x \leq 4]\!] \mid\mid \mathcal{I}[\![x \leq 5]\!]$

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$\Rightarrow$ **Solution**: Use atomic load and store!

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Propagators:

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Propagators:

$$\begin{array}{llll} & x \leftarrow [-\infty, 4] & & (\mathcal{I}[\![x \leq 4]\!]) \\ \parallel & x \leftarrow [-\infty, 5] & & (\mathcal{I}[\![x \leq 5]\!]) \end{array}$$

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Propagators:

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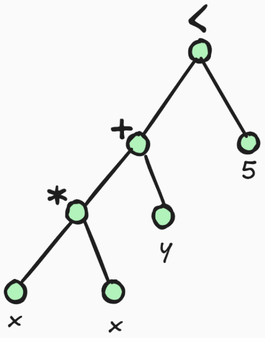
Issue 3: progress? What if $\mathcal{I}[\![x \leq 5]\!]$ is always "winning"?

$\Rightarrow$ **Solution**: Write in the memory only if the value is strictly lower ($\lceil x \rceil = v$ iff $v < \lceil x \rceil$).

Ternary Constraint Network

Representation of Propagators



- Represented using `shared_ptr` and `variant` data structures.

  $\Rightarrow$ Uncoalesced memory accesses.

- Code similar to an interpreter:

```
switch(term.index()) {
  case IVar:
  case INeg:
  case IAdd:
  case IMul:
  // ...
```

  $\Rightarrow$ Thread divergence.

Ternary Normal Form (TNF)

We simplify the representation of constraints to ternary constraints of the form:

- `x = y <op> z` where `x,y,z` are variables.
- The operators are $\{+, /, *, \mathit{mod}, \mathit{min}, \mathit{max}, \leq, =\}$.

Example

The constraint $x + y \neq 2$ is represented by:

```
t1 = x + y
ZERO = (t1 = TWO)
```

`equivalent to` *false* $\Leftrightarrow (t1 = 2)$

where `ZERO` and `TWO` are two variables with constant values.

Bytecode Representation

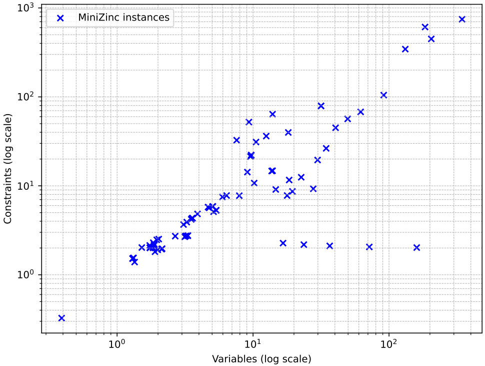
The ternary form of a propagator holds on 16 bytes:

```
struct bytecode_type {
  int op;
  int x;
  int y;
  int z;
};
```

- **Uniform representation** of propagators in memory $\Rightarrow$ coalesced memory accesses.
- Limited number of operators + sorting $\Rightarrow$ reduced thread divergence.

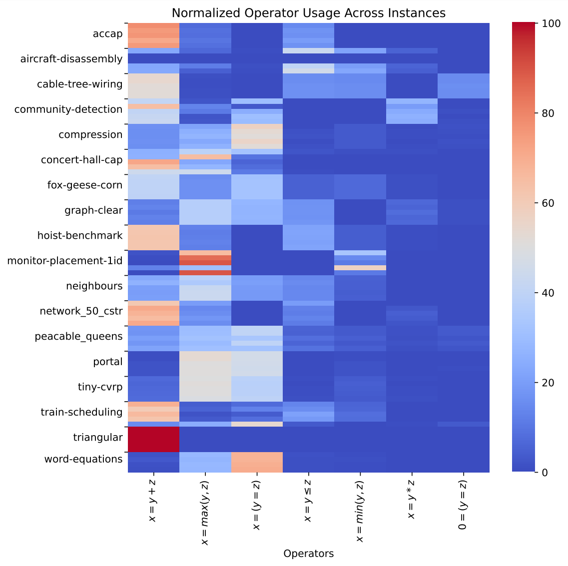
Drawback of TNF: increase in number of propagators and variables.

Benchmark on the MiniZinc Challenge 2024 (89 instances)



The **median increase** of variables is 4.76x and propagators is 4.33x.

Divergence?



Normalized Operator Usage Across Instances
accap
aircraft-disassembly
cable-tree-wiring
community-detection
compression
concert-hall-cap
fox-geese-corn
graph-clear
hoist-benchmark
monitor-placement-1id
neighbours
network_50_cstr
peacable_queens
portal
tiny-cvrp
train-scheduling
triangular
word-equations
x = y + z
x = max(y, z)
x = (y = z)
x = y ≤ z
x = min(y, z)
x = y * z
0 = (y = z)
Operators
100
80
60
40
20
0

Benchmarks: Turbo vs Choco v4.18

Comparison of the best objective values found (timeout: 20 mins, GPU: H100).