UNIVERSITÉ DU LUXEMBOURG

Lecture 1: Constraint Network

Intelligent Systems—Problem Solving
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Goals

- ✯ Understanding a formal definition of constraint network.
- ✯ Manipulate the mathematical concept of constraint network.
- ✯ Modelling a simple constraint problem.

1 Constraint Network

In the following, we consider constraint programming over integer variables only. Let X be a finite set of variables. We denote $\mathbf{C}$ the set of all constraints generated by the rule $\langle c_1, c_2 \rangle$ of the following grammar:

$\langle \odot \rangle$ ::= + | - | $\times$ | $tdiv$

$\langle \diamond \rangle$ ::= $=$ | $\neq$ | $<$ | $>$ | $\geq$ | $\leq$

$\langle t, u \rangle$ ::=	*definition of terms*
| x	*variable* $x \in X$
| c	*constant* $c \in \mathbb{Z}$
| $t \odot u$	*functions*
| (t)	*grouping*
$\langle c_1, c_2 \rangle$ ::=	*formulas*
| $t \diamond u$	*constraints*
| $\neg c_1$	*negation*
| $c_1 \land c_2$	*conjunction*
| $c_1 \lor c_2$	*disjunction*
| $c_1 \Rightarrow c_2$	*implication*
| $c_1 \Leftrightarrow c_2$	*equivalence*
| (c_1)	*grouping*

Note that $tdiv$ is the truncated division, for instance $5 \; tdiv \; 2 = 2$ and $-5 \; tdiv \; 2 = -2$.

Exercise 1 – Grammar of constraints

- Is $x \leq y \times z$ a constraint of the grammar above?
- Is $x + y = z \ tdiv \ 2$ a constraint of the grammar above?
- Is $x + y$ a constraint of the grammar above?
- Is $2 + 2 = 4$ a constraint of the grammar above?
- Is $2 + 2 = 3$ a constraint of the grammar above?
- Is $x \geq 5 \land x \leq 10$ a constraint of the grammar above?
- Is $\neg(x \geq 5 \land x \leq 10)$ a constraint of the grammar above?

end of exercise.

For each constraint $c \in \mathbf{C}$, we denote by $scp(c) \subseteq X$ its *scope*, i.e. the set of free variables of c.

Exercise 2 – Scope of a constraint

What is the scope of:

- $scp(x \neq 5) =$
- $scp(x + y \leq z) =$
- $scp(1 \neq 5) =$

end of exercise.

An *assignment* is a map $asn : X \to \mathbb{Z}$, and we denote the set of all assignments by $\mathbf{Asn}$. For each constraint $c \in \mathbf{C}$, we define a function $rel \in \mathbf{C} \to \mathcal{P}(\mathbf{Asn})$ defining the set of solutions of this constraint. For instance, $rel(x + y \leq 5) \triangleq \{asn \in \mathbf{Asn} \mid asn(x) + asn(y) \leq 5\}$.

Exercise 3 – Solutions of constraints

Give the definition of rel for the following constraints:

- $rel(x = 1) \triangleq$
- $rel(x = y) \triangleq$
- $rel(\neg(x \leq y)) \triangleq$
- $rel(x < 10 \lor x > 20) \triangleq$

end of exercise.

A *constraint network* is a pair $P = \langle d, C \rangle$ such that $d \in X \to \mathcal{P}(\mathbb{Z})$ is the *domain function* and $C \subseteq \mathbf{C}$ is a finite set of constraints. We denote $\mathbf{D}$ the set of all domain functions $X \to \mathcal{P}(\mathbb{Z})$ ordered pointwise ($d \leq d' \Leftrightarrow \forall x \in X,\ d(x) \subseteq d'(x)$). The set of solutions of a constraint network is:

$$sol(d, C) \triangleq \{asn \in \mathbf{Asn} \mid \forall c \in C,\ asn \in rel(c) \land \forall x \in X,\ asn(x) \in d(x)\}$$

Finding an assignment $s \in sol(d, C)$ is the *constraint satisfaction problem*.

Exercise 4 – Solutions of constraint networks

- Let $P = \langle \{x \mapsto \{0,1,2\}, y \mapsto \{2,3,4\}\}, \{x \neq y, x > 1\} \rangle$ be a constraint network.
  Compute $sol(P) =$

- Let $P = \langle \{x \mapsto \{0,1\}, y \mapsto \{0,1\}, z \mapsto \{0,1\}\}, \{x \neq y, y \neq z, z \neq x\} \rangle$ be a constraint network.
  Compute $sol(P) =$

end of exercise.

Exercise 5 – Modelling Sudoku

Describe a constraint network $\langle d, C \rangle$ such that $sol(d, C)$ represents all the solutions of a partially completed grid of Sudoku. Give an integer linear programming (ILP) formulation of this problem and compare both models.

end of exercise.

Exercise 6 – Alternative definition of sol

Is the following definition of sol', equivalent to sol? Prove it, or disprove it.

$$sol'(d, C) \triangleq \bigcap_{c \in C} rel(c) \cap \{asn \in \mathbf{Asn} \mid \forall x \in X,\ asn(x) \in d(x)\}$$

end of exercise.

Exercise 7 – Redundant constraints

Let $\langle d, C \rangle$ be a constraint network. A redundant constraint $c \in C$ is a constraint that is not necessary—removing it does not change the set of solutions of our problem. Formally define what is a redundant constraint.

end of exercise.